Probability & Statistics (1)

Jointly Distributed Random Variables (II)

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Sums of Independent Random Variables
Uniform Distribution

- 時常我們會需要用到兩個獨立的隨機變數相加的機率分配，譬如說現在有兩個隨機變數: X與Y。我們需要計算$X+Y$的$c.d.f.$就可以從它們各自的$p.d.f.(f_X \text{ and } f_Y)$計算得知:

$$F_{X+Y}(a) = P\{X+Y \le a\} = \iint_{x+y\le a} f_X(x)f_Y(y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{a-y} f_X(x)dxf_Y(y)dy$$

$$= \int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy$$

F_{X+Y}的$c.d.f.$其實也就是F_X與F_Y的convolution。

Sums of Independent Random Variables
Uniform Distribution

Differentiation both side,

$$f_{X+Y}(a) = \frac{a}{da}\int_{-\infty}^{\infty} F_X(a-y)f_Y(y)dy = \int_{-\infty}^{\infty} \frac{d}{da}F_X(a-y)f_Y(y)dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y)f_Y(y)dy$$

Sums of Independent Random Variables
Uniform Distribution

- **範例一**

如果X與Y為獨立隨機變數且值域uniformly distributed on (0,1)，試問$X+Y$的$p.d.f.$。

Solution:

Let f_X and f_Y are the $p.d.f.$ of X and Y, respectively. Let $Z=X+Y$.

$$f_Z(z)=\int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx\,, where\ 0<x<1\ and\ 0<z-x<1$$

$$f_Z(z)=0, for\ z<0\ and\ z\geq 2; in\ other\ words, 0<z<2$$

Sums of Independent Random Variables
Uniform Distribution

(1) $\boldsymbol{Case\ 1: 0 < z \leq 1, f_X(x) f_Y(z-x)}$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$$

$To\ confirm\ f_Y(z-x) = 1, we\ need\ z - x \geq 0 \Rightarrow x \leq z$

$$\int_0^z 1 dx = z\,; f_Z(z) = z, for\ 0 < z \leq 1$$

(2) $\boldsymbol{Case\ 2: 1 < z < 2}$

$To\ confirm\ f_Y(z-x) = 1, we\ need\ z - x \leq 1 \Rightarrow x \geq z - 1$

$$\int_{z-1}^1 1 dx = 2 - z\,; f_Z(z) = 2 - z, for\ 1 < z < 2$$

Sums of Independent Random Variables
Uniform Distribution

As a result,

$$f_Z(z) = f_{X+Y}(z) = \begin{cases} z, 0 \leq z \leq 1 \\ 2 - z, 1 < z < 2 \\ 0, otherwise \end{cases}$$

Because of the shape of $p.d.f.$, the random variable of $X + Y$ is a $triangular\ distribution$.

Sums of Independent Random Variables
Uniform Distribution

假設$X_1, X_2, \ldots, X_n$為independent uniform random variable，使得

$$F_n(x) = P\{X_1 + \cdots + X_n \leq x\}$$

雖然$F_n(x)$的general form非常難表示，不過$x \leq 1$的時候，就可以用mathematical induction得到

$$F_n(x) = \frac{x^n}{n!}, 0 \leq x \leq 1$$

已知$n = 1$是正確的，假設

$$F_{n-1}(x) = \frac{x^{n-1}}{(n-1)!}, 0 \leq x \leq 1$$

Sums of Independent Random Variables
Uniform Distribution

於是就可以表示成

$$\sum_{i=1}^{n} X_i = \sum_{i=1}^{n-1} X_i + X_n, X_i \; is \; nonnegative, where \; 0 \le x \le 1$$

$$F_n(x) = \int_0^1 F_{n-1}(x-y) f_{X_n}(y)dy = \frac{1}{(n-1)!}\int_0^x (x-y)^{n-1}dy = \frac{x^n}{n!}$$

$for \; n = 1, X \; follows \; a \; uniform \; distribution$

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

$for \; n = 2, X \; follows \; a \; triangular \; distribution$

$$f_X(x) = \begin{cases} x & 0 \le x \le 1 \\ 2 - x & 1 \le x \le 2 \end{cases}$$

$for \; n = 3,$

$$f_X(x) = \begin{cases} \frac{1}{2}x^2 & 0 \le x \le 1 \\ \frac{1}{2}(-2x^2 + 6x - 3) & 1 \le x \le 2 \\ \frac{1}{2}(3-x)^2 & 2 \le x \le 3 \end{cases}$$

Sums of Independent Random Variables
Uniform Distribution

$for\ n = 4,$

$$f_X(x) = \begin{cases} \frac{1}{6}x^3 & 0 \le x \le 1 \\ \frac{1}{6}(-3x^3 + 12x^2 - 12x + 4) & 1 \le x \le 2 \\ \frac{1}{6}(3x^3 - 24x^2 + 60x - 44) & 2 \le x \le 3 \\ \frac{1}{6}(4 - x)^3 & 3 \le x \le 4 \end{cases}$$

Sums of Independent Random Variables
Uniform Distribution

$for\ n = 5,$

$$f_X(x) = \begin{cases} \frac{1}{24}x^4 & 0 \le x \le 1 \\ \frac{1}{24}(-4x^4 + 20x^3 - 30x^2 + 20x - 5) & 1 \le x \le 2 \\ \frac{1}{24}(6x^4 - 60x^3 + 210x^2 - 300x + 155) & 2 \le x \le 3 \\ \frac{1}{24}(-4x^4 + 60x^3 - 330x^2 + 780x - 655)^3 & 3 \le x \le 4 \\ \frac{1}{24}(5 - x)^4 & 4 \le x \le 5 \end{cases}$$

Sums of Independent Random Variables
Normal Distribution

Proposition 1

If X_i, $i = 1, \ldots, n$, are independent random variables that are normally distributed with respective parameters μ_i, $\sigma_i^2, i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i$ is normally distributed with parameters $\sum_{i=1}^{n} \mu_i$ and $\sum_{i=1}^{n} \sigma_i^2$.

Proof:

Let X and Y be independent normal random variables with X having mean 0 and variance σ^2 and Y having mean 0 and variance 1. We will determine the density function of $X + Y$ by utilizing $f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y) dy$.

Sums of Independent Random Variables
Normal Distribution

For independent random variables X and Y, the distribution f_Z of $Z = X + Y$ equals the convolution of f_X and f_Y:

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(z - x) f_X(x) dx$$

Given that f_X and f_Y are normal densities,

$$f_X(x) = \mathcal{N}(x; \mu_X, \sigma_X^2) = \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left\{-\frac{(x - \mu_X)^2}{2\sigma_X^2}\right\}$$

$$f_Y(y) = \mathcal{N}(y; \mu_Y, \sigma_Y^2) = \frac{1}{\sigma_Y \sqrt{2\pi}} \exp\left\{-\frac{(y - \mu_Y)^2}{2\sigma_Y^2}\right\}$$

Sums of Independent Random Variables
Normal Distribution

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}\right] \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{\sigma_X^2(z-x-\mu_Y)^2 + \sigma_Y^2(x-\mu_X)^2}{2\sigma_X^2\sigma_Y^2}\right] dx$$

$$= \int_{\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{\sigma_X^2(z^2+x^2+\mu_Y^2-2xz-2z\mu_Y+2x\mu_Y) \quad + \sigma_Y^2(x^2+\mu_X^2-2x\mu_X)}{2\sigma_X^2\sigma_Y^2}\right] dx$$

$$= \int_{\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{x^2(\sigma_X^2+\sigma_Y^2) - 2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X) + \sigma_X^2(z^2+\mu_Y^2-2z\mu_Y) + \sigma_Y^2\mu_X^2}{2\sigma_X^2\sigma_Y^2}\right] dx$$

Let $\sigma_Z = \sqrt{\sigma_X^2 + \sigma_Y^2}$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sigma_Z\sqrt{2\pi}} \frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}} \exp\left[-\frac{\frac{x^2(\sigma_Z^2)}{\sigma_Z^2} - 2x\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2} + \frac{\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

Sums of Independent Random Variables
Normal Distribution

$$f_Z(z)=\int_{-\infty}^{\infty}\frac{1}{\sigma_Z\sqrt{2\pi}}\frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}}\exp\left[-\frac{x^2-2x\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}+\frac{\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right]dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma_Z\sqrt{2\pi}}\frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}}\exp\left[-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2-\left(\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2+\frac{\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right]dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma_Z\sqrt{2\pi}}\exp\left[-\frac{-(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X)^2+\sigma_Z^2(\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2)}{2\sigma_Z^2(\sigma_X\sigma_Y)^2}\right]\frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}}\exp\left[\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right]dx$$

$$=\frac{1}{\sigma_Z\sqrt{2\pi}}\exp\left[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}\right]\int_{-\infty}^{\infty}\frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}}\exp\left[\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right]dx$$

Sums of Independent Random Variables
Normal Distribution

$$f_Z(z) = \frac{1}{\sigma_Z\sqrt{2\pi}} \exp\left[-\frac{\left(z-(\mu_X+\mu_Y)\right)^2}{2\sigma_Z^2}\right] \int_{-\infty}^{\infty} \frac{1}{\frac{\sigma_X\sigma_Y}{\sigma_Z}\sqrt{2\pi}} \exp\left[\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

The expression in the integral is a normal density distribution on x, and so the integral evaluates to 1. Therefore, …

$$f_Z(z) = \frac{1}{\sigma_Z\sqrt{2\pi}} \exp\left[-\frac{\left(z-(\mu_X+\mu_Y)\right)^2}{2\sigma_Z^2}\right]$$

Sums of Independent Random Variables
Normal Distribution

- **範例二**

假設籃球比賽一季有44場比賽: 26場與一級球隊比賽；18場與二級球隊比賽。假設與一級球隊比賽的勝率為0.4，而與二級球隊比賽的勝率為0.7，而每一場比賽接為獨立事件，則:

(1) 某一球隊贏25場的機率為何?

(2) 某一球隊贏一級球隊多於二級球隊的機率為何?

Sums of Independent Random Variables
Normal Distribution

Solution:

(1)某一球隊贏25場的機率為何?

令X_A與X_B分別為贏一級球隊與二級球隊的比賽場數，且X_A與X_B為independent binomial random variable。

$$E[X_A]=26\times0.4=10.4; Var(X_A)=26\times0.4\times0.6=6.24$$
$$E[X_B]=18\times0.7=12.6; Var(X_B)=18\times0.7\times0.3=3.78$$

此時，我們可以用normal approximation來逼近binomial，X_A與X_B會逼近normal distribution相同參數(μ,σ^2)的分布。

依據**Proposition 1**可以得知，X_A+X_B所逼近的常態分配參數為$(\mu=10.4+12.6=23,\sigma^2=6.24+3.78=10.02)$

Sums of Independent Random Variables
Normal Distribution

$$P\{X_A + X_B \geq 25\} = P\{X_A + X_B \geq 24.5\}$$

$$= P\left\{\frac{X_A + X_B - 23}{\sqrt{10.02}} \geq \frac{24.5 - 23}{\sqrt{10.02}}\right\} \approx P\left\{Z \geq \frac{1.5}{\sqrt{10.02}}\right\}$$

$$\approx 1 - P\{Z < 0.4739\} \approx 0.3178$$

Sums of Independent Random Variables
Normal Distribution

(2)某一球隊贏一級球隊多於二級球隊的機率為何?

此提要求的就是 $P\{X_A - X_B \geq 1\}$，對於 $X_A - X_B$ 來說，其 normal distribution 的參數為 $(\mu = 10.4 - 12.6 = -2.2, \sigma^2 = 6.24 + 3.78 = 10.02)$

$$\because Var(X - Y) = Var\big(X + (-Y)\big) = Var(X) + Var(-Y),$$

$$since\ Var(-Y) = Var(Y)$$

$$\therefore Var(X - Y) = Var(X) + Var(Y)$$

$$P\{X_A - X_B \geq 1\} = P\{X_A + X_B \geq 0.5\}$$

$$= P\left\{\frac{X_A - X_B - (-2.2)}{\sqrt{10.02}} \geq \frac{0.5 - (-2.2)}{\sqrt{10.02}}\right\} \approx P\left\{Z \geq \frac{2.7}{\sqrt{10.02}}\right\}$$

$$\approx 1 - P\{Z < 0.8530\} \approx 0.1968$$

Sums of Independent Random Variables
Lognormal Distribution

如果$\log(Y)$為一個normal random variable (μ, σ^2)，則Y就可以被定義為一個lognormal random variable (μ, σ)。Y可以被表示為

$$Y = e^X$$

where X is a normal random variable

Sums of Independent Random Variables
Normal Distribution

- **範例三**

從某個固定時間開始，令$S(n)$表示在 n 週結束時某種證券的價格, $n \geq 1$。常見的價格預測方式為價格比$S(n)/S(n-1)$, $n \geq 1$，為 independent and identical distributed lognormal random variables。假設其$\mu = 0.0165$，$\sigma = 0.0730$，是求以下的機率:

(1) 證券價格在未來兩周內都會增長

(2) 兩週後的價格高於今天的價格

Sums of Independent Random Variables
Normal Distribution

Solution:

(1) 證券價格在未來兩周內都會增長

令Z為standard normal random variable，為了解第一題，我們可以利用題目的線索，也就是持續增長，亦即$\log(x)$必須隨著x增加而增加，所以x必須大於1，也可以表示為$\log(x) > \log(1) \Rightarrow \log(x) - \log(1) > 0$。

$$P\left\{\frac{S(1)}{S(0)} > 1\right\} = P\left\{\log\left(\frac{S(1)}{S(0)}\right) > 0\right\} = P\left\{Z > \frac{0 - 0.0165}{0.0730}\right\}$$

$$= P\{Z < 0.2260\} = 0.5894; \textit{each of next two weeks} = (0.5894)^2$$

$$= 0.3474$$

Sums of Independent Random Variables
Normal Distribution

(2)兩週後的價格高於今天的價格

$$P\left\{\frac{S(2)}{S(0)} > 1\right\} = P\left\{\frac{S(2)}{S(1)}\frac{S(1)}{S(0)} > 1\right\} = P\left\{\log\left(\frac{S(2)}{S(1)}\right) + \log\left(\frac{S(1)}{S(0)}\right) > 0\right\}$$

這就可以視為兩個independent normal random variable相加，所以相加後的normal random variable ($\mu = 0.0165 + 0.0165 = 0.0330, \sigma^2 = 2 \times (0.0730)^2$)

$$P\left\{\frac{S(2)}{S(0)} > 1\right\} = P\left\{Z > \frac{0 - 0.0330}{0.0730\sqrt{2}}\right\} = P\{Z < 0.31965\} = 0.6254$$

Sums of Independent Random Variables
Poisson and Binomial Distribution

- **範例四**

如果X與Y為independent Poisson random variable，其參數分別為λ_1與λ_2，是求出$X+Y$的$p.m.f.$:

Solution:

我們可以假設$\{X+Y=n\}$，也可以拆成$\{X=k, Y=n-k\}, 0 \le k \le n$，則

$$P\{X+Y=n\} = \sum_{k=0}^{n} P\{X=k, Y=n-k\} = \sum_{k=0}^{n} P\{X=k\}P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!} e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)}\sum_{k=0}^{n}\frac{\lambda_1^k\lambda_2^{n-k}}{k!\,(n-k)!}$$

Sums of Independent Random Variables
Poisson and Binomial Distribution

$$= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)} \sum_{k=0}^{n} \frac{\lambda_1^k \lambda_2^{n-k}}{k!\,(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \lambda_1^k \lambda_2^{n-k} = \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n$$

故$X + Y$為一個Poisson distribution $(\lambda_1 + \lambda_2)$

Sums of Independent Random Variables
Poisson and Binomial Distribution

- 範例五

如果X與Y為independent binomial random variables，其參數分別為(n,p)與(m,p)，試問$X+Y$的$p.m.f.$為何?

Solution:

$X+Y$的binomial random variable的參數為$(n+m,p)$，故:

$$P\{X+Y\}=\sum_{i=0}^{n} P\{X=i,Y=k-i\}=\sum_{i=0}^{n} P\{X=i\}P\{Y=k-i\}$$

$$=\sum_{i=0}^{n}\binom{n}{i}p^{i}q^{n-i}\binom{m}{k-i}p^{k-i}q^{m-k+i},$$

Sums of Independent Random Variables
Poisson and Binomial Distribution

$$P\{X+Y\} = \sum_{i=0}^{n} \binom{n}{i} p^i q^{n-i} \binom{m}{k-i} p^{k-i} q^{m-k+i}$$

$$= p^{i+k-i} q^{n-i+m-k+i} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i} = p^k q^{n+m-k} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i}$$

$$\Rightarrow P\{X+Y\} = p^k q^{n+m-k} \sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i}$$

$since\ P\{X+Y\} \sim binomial(n+m,k), thus \Rightarrow$

$$\sum_{i=0}^{n} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$$

Sums of Independent Random Variables
Geometric Distribution

- **範例六**

令$X_1, \dots, X_n$為independent geometric random variables，且其機率為$p_i\ for\ i = 1, \dots, n$。試問$S_n = \sum_{i=1}^{n} X_i$的$p.m.f.$為何?

Solution:

我們可以試想一個投擲硬幣的情境，X_n就可以想做有n枚硬幣，針對第i枚硬幣出現正面的機率為p_i，於是乎$S_n = \sum_{i=1}^{n} X_i$就可以被當作投擲了n次正面的機率，所以就可以被視為negative binomial random variable。

$$P\{S_n = k\} = \binom{k-1}{n-1} p^n (1-p)^{k-n}, k \geq n$$

Sums of Independent Random Variables
Geometric Distribution

$$P\{S_n = k\} = \binom{k-1}{n-1} p^n (1-p)^{k-n}, k \geq n$$

為了歸納其S_n 的$p.m.f.$，假設所有的p_i都不一樣。我們先考慮$n = 2$的情形，令$q_j = 1 - p_j, j = 1,2$，則

$$P\{S_2 = k\} = \sum_{j=1}^{k-1} P\{X_1 = j, X_2 = k - j\} = \sum_{j=1}^{k-1} P\{X_1 = j\} P\{X_2 = k - j\}$$

$$= \sum_{j=1}^{k-1} p_1 q_1^{j-1} p_2 q_2^{k-j-1} = p_1 p_2 q_2^{k-2} \sum_{j=1}^{k-1} \left(\frac{q_1}{q_2}\right)^{j-1}$$

$$= p_1 p_2 q_2^{k-2} \frac{1 - (q_1/q_2)^{k-1}}{1 - q_1/q_2} = p_1 p_2 q_2^{k-2} q_2 \frac{1 - (q_1/q_2)^{k-1}}{q_2 - q_1}$$

Sums of Independent Random Variables
Geometric Distribution

$$P\{S_2 = k\} = p_1 p_2 q_2^{k-2} q_2 \frac{1-(q_1/q_2)^{k-1}}{q_2 - q_1} = p_1 p_2 q_2^{k-1} \frac{1-(q_1/q_2)^{k-1}}{q_2 - q_1}$$

$$= \frac{p_1 p_2 q_2^{k-1}}{q_2 - q_1} - \frac{p_1 p_2 q_2^{k-1} \left(\frac{q_1}{q_2}\right)^{k-1}}{q_2 - q_1} = \frac{p_1 p_2 q_2^{k-1}}{q_2 - q_1} - \frac{p_1 p_2 q_1^{k-1}}{q_2 - q_1}$$

$$= p_2 q_2^{k-1} \frac{p_1}{q_2 - q_1} - p_1 q_1^{k-1} \frac{p_2}{q_2 - q_1}$$

$$= p_2 q_2^{k-1} \frac{p_1}{p_1 - p_2} - p_1 q_1^{k-1} \frac{p_2}{p_1 - p_2}$$

$$= p_2 q_2^{k-1} \frac{p_1}{p_1 - p_2} + p_1 q_1^{k-1} \frac{p_2}{p_2 - p_1}$$

Sums of Independent Random Variables
Geometric Distribution

那如果是$n=3$的時候，$P\{S_3=k\}$，

$$P\{S_3=k\}=\sum_{j=1}^{k-1}P\{S_2=j,X_3=k-j\}=\sum_{j=1}^{k-1}P\{S_2=j\}P\{X_3=k-j\}$$

$$=p_1q_1^{k-1}\frac{p_2}{p_2-p_1}\frac{p_3}{p_3-p_1}+p_2q_2^{k-1}\frac{p_1}{p_1-p_2}\frac{p_3}{p_3-p_2}+p_3q_3^{k-1}\frac{p_1}{p_1-p_3}\frac{p_2}{p_2-p_3}$$

Sums of Independent Random Variables
Geometric Distribution

根據剛剛的規律，我們可以得出一個新的proposition。

- **Proposition 2**

Let $X_1, \ldots, X_n$ 為 independent geometric random variables, with X_i having parameters p_i, for $i = 1, \ldots, n$. If all the p_i are distinct, then , for $k \geq n$,

$$P\{S_n = k\} = \sum_{i=1}^{n} p_i q_i^{k-1} \prod_{j \neq i} \frac{p_j}{p_j - p_i}$$

Conditional Distributions: Discrete Case

- 給定任何兩個事件E與F，則F成立前提下E所發生的條件機率(conditional probability)為:

$$P(E|F) = \frac{P(EF)}{P(F)}, where\ P(F) > 0$$

- 當今天有兩個discrete random variable X與Y，在Y成立前提下X發生的$conditional\ probability\ mass\ function$為

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}} = \frac{p(x,y)}{p_Y(y)},$$

$$where\ p_Y > 0$$

Conditional Distributions: Discrete Case

- 那計算在Y成立前提下X發生的$cumulative\ conditional\ p.m.f.$為

$$F_{X|Y}(x|y) = P\{X \le x, Y \le y\} = \sum_{a \le x} p_{X|Y}(a|y)$$

- 如果X與Y相互獨立的話，

$$p_{X|Y}(x|y) = P\{X = x|Y = y\} = \frac{P\{X = x, Y = y\}}{P\{Y = y\}} = \frac{P\{X = x\}P\{Y = y\}}{P\{Y = y\}}$$

$$= P\{X = x\}$$

Conditional Distributions: Discrete Case

- **範例七**

令$p(x,y)$為X與Y的$joint\ p.m.f.$，

$$p(0,0)=0.4; p(0,1)=0.2; p(1,0)=0.1; p(1,1)=0.3$$

請問$Y=1$的前提下X發生的$conditional\ p.m.f.$為何?

Solution:

$$p_Y(1)=\sum_x p(x,1)=p(0,1)+p(1,1)=0.2+0.3=0.5$$

$$p_{X|Y}(0|1)=\frac{p(0,1)}{p_Y(1)}=\frac{2}{5};\ p_{X|Y}(1|1)=\frac{p(1,1)}{p_Y(1)}=\frac{3}{5}$$

Conditional Distributions: Discrete Case

- **範例八**

假設X與Y皆為independent Poisson random variables，其參數分別為λ_1與λ_2，試問在$X+Y=n$的前提下X的conditional probability為何?

Solution:

本題要計算$X+Y=n$前提下X的conditional probability mass function。

$$P\{X=k \mid X+Y=n\}=\frac{P\{X=k, X+Y=n\}}{P\{X+Y=n\}}=\frac{P\{X=k, Y=n-k\}}{P\{X+Y=n\}}$$

$$=\frac{P\{X=k\}P\{Y=n-k\}}{P\{X+Y=n\}}=\frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1}$$

Conditional Distributions: Discrete Case

$$P\{X = k \mid X + Y = n\} = \frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}\left[\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}\right]^{-1}$$

$$= \frac{n!}{(n-k)!\,k!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n} = \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{n-k}$$

The conditional distribution of X given that $X + Y = n$ is the binomial distribution with parameters n and $\frac{\lambda_1}{\lambda_1+\lambda_2}$

Conditional Distributions: Continuous Case

- 如果X與Y的joint probability density function為$f(x,y)$，則在給定$Y=y$的前提下X的conditional probability density function可以被定義，對於每一個y其$f_Y(y)>0$，

$$f_{X+Y}(x|y)dx=\frac{f(x,y)dxdy}{f_Y(y)dy}\approx\frac{P\{x\le X\le x+dx,y\le Y\le y+dy\}}{P\{y\le y\le y+dy\}}$$

$$=P\{x\le X\le x+dx|y\le Y\le y+dy\}$$

- 如果X與Y是jointly continuous random variable，則對於任何集合A

$$P\{X\in A|Y=y\}=\int_A f_{X|Y}(x|y)dx$$

Conditional Distributions: Continuous Case

- Let $A = (-\infty, a]$，則在給定$Y = y$前提下X的conditional cumulative distribution function為

$$F_{X+Y}(a|y) \equiv P\{X \leq a|Y = y\} = \int_{-\infty}^{a} f_{X+Y}(x|y)dx$$

Conditional Distributions: Continuous Case

- 範例九

X與Y的joint density function為

$$f(x,y)=\begin{cases}\frac{12}{5}x(2-x-y) & 0<x<1, 0<y<1\\ 0 & otherwise\end{cases}$$

計算$Y=y$前提下X的conditional density為何?

Solution:

$$f_{X+Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f(x,y)}{\int_{-\infty}^{\infty}f(x,y)dx}=\frac{x(2-x-y)}{\int_0^1 x(2-x-y)dx}=\frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}}$$

$$=\frac{6x(2-x-y)}{4-3y}$$

Conditional Distributions: Continuous Case

- **範例十**

假設X與Y的joint density function為

$$f(x,y)=\begin{cases}\dfrac{e^{-\frac{x}{y}}e^{-y}}{y} & 0<x<\infty, 0<y<\infty\\ 0 & otherwise\end{cases}$$

試問$P\{X>1|Y=y\}$。

Solution:

$$f_{X+Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{\dfrac{e^{-\frac{x}{y}}e^{-y}}{y}}{e^{-y}\int_0^{\infty}\frac{1}{y}e^{-\frac{x}{y}}dx}=\frac{1}{y}e^{-\frac{x}{y}}$$

Conditional Distributions: Continuous Case

$$f_{X+Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{e^{-\frac{x}{y}}e^{-y}}{y}}{e^{-y}\int_0^\infty \frac{1}{y}e^{-\frac{x}{y}}dx} = \frac{1}{y}e^{-\frac{x}{y}}$$

Hence,

$$P\{X > 1|Y = y\} = \int_1^\infty \frac{1}{y}e^{-\frac{x}{y}}dx = -e^{-\frac{x}{y}}\Big|_1^\infty = e^{-\frac{1}{y}}$$

Conditional Distributions: Continuous Case

If X and Y are independent continuous random variables, the conditional density of X given that $Y = y$ is just the unconditional density of X. This is so because, in the independent case,

$$f_{X+Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$

Order Statistics

- Let $X_1, X_2, \dots, X_n$ be n independent and identically distributed continuous random variables having a common density f and distribution function F. Define

$$X_{(1)} = smallest\ of\ X_1, X_2, \dots, X_n$$
$$X_{(2)} = second\ smallest\ of\ X_1, X_2, \dots, X_n$$
$$\vdots$$
$$X_{(j)} = jth\ smallest\ of\ X_1, X_2, \dots, X_n$$
$$\vdots$$
$$X_{(n)} = largest\ of\ X_1, X_2, \dots, X_n$$

Order Statistics

- The ordered values $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are known as the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_n$. In other words, $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the ordered values of $X_1, X_2, \ldots, X_n$.
- The joint density function of the ordered statistics is obtained by noting that the order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ will take on the values $x_1 \leq x_2 \leq \cdots \leq x_n$ if and only if, for some permutation $(i_1, i_2, \ldots, i_n)$ of $(1,2, \ldots, n)$,

$$X_1 = x_{i_1}, X_2 = x_{i_2}, \ldots, X_n = x_{i_n}$$

Order Statistics

- Since, for any permutation $(i_1, i_2, \dots, i_n)$ of $(1,2,\dots,n)$,

$$P\left\{x_{i_1} - \frac{\varepsilon}{2} < X_1 < x_{i_1} + \frac{\varepsilon}{2}, \dots, x_{i_n} - \frac{\varepsilon}{2} < X_n < x_{i_n} + \frac{\varepsilon}{2}\right\}$$

$$\approx \varepsilon^n f_{X_1,\dots X_n}\left(x_{i_1}, \dots x_{i_n}\right) = \varepsilon^n f\left(x_{i_1}\right) \dots f\left(x_{i_n}\right) = \varepsilon^n f(x_i) \dots f(x_n)$$

It follows that, for $x_1 < x_2 < \cdots < x_n$,

$$P\left\{x_1 - \frac{\varepsilon}{2} < X_{(1)} < x_1 + \frac{\varepsilon}{2}, \dots, x_n - \frac{\varepsilon}{2} < X_{(n)} < x_n + \frac{\varepsilon}{2}\right\}$$

$$\approx n!\, \varepsilon^n f(x_1) \dots f(x_n)$$

Dividing by ε^n and letting $\varepsilon \to 0$ yields

$$\varepsilon^n f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots x_n) = n!\, \varepsilon^n f(x_1) \dots f(x_n)$$

$$f_{X_{(1)},\dots,X_{(n)}}(x_1, \dots x_n) = n!\, f(x_1) \dots f(x_n), x_1 < x_2 < \cdots < x_n$$

Order Statistics

- **範例十一**

有3個人隨機位在(distributed at random)一條1公里的道路上，試問: 不存在任兩人相距小於d公里($d < \frac{1}{2}$)的機率。

Solution:

distributed at random這句話可以當作independent and uniformly distributed。令X_i為第i個人在這條道路上的位置，我們要求的機率就可以表示為$P\{X_{(i)} > X_{(i-1)} + d, i = 2,3\}$

$$f_{X_{(1)},X_{(2)},X_{(3)}}(x_1, x_2, x_3) = 3!, 0 < x_1 < x_2 < x_3 < 1$$

Order Statistics

$$P\{X_{(i)} > X_{(i-1)} + d, i = 2,3\} = \iiint_{x_i > x_{j-1}+d} f_{X_{(1)},X_{(2)},X_{(3)}}(x_1, x_2, x_3)\, dx_1 dx_2 dx_3$$

$$= 3! \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} dx_3 dx_2 dx_1 = 6 \int_0^{1-2d} \int_{x_1+d}^{1-d} (1 - x_2 - d) dx_2 dx_1$$

$$let\ (1 - x_2 - d) = y_2$$

$$= 6 \int_0^{1-2d} \int_0^{1-d-(x_1+d)=1-2d-x_1} y_2 dy_2 dx_1 = 3 \int_0^{1-2d} (1 - 2d - x_1)^2 dx_1$$

$$= 3 \int_0^{1-2d} y_1^2 dy_1 = (1 - 2d)^3 \Rightarrow if\ there\ are\ n\ people, \ldots [1 - (n-1)d]^n, where\ d < \frac{1}{n-1}$$

Order Statistics

- The density function of the $j-th$-order statistics $X_{(j)}$ can be obtained either by integrating the joint density function (pp.49) or by direct reasoning as follows: In order for $X_{(j)}$ to equal x, it is necessary for $j-1$ of the n values $X_1, \ldots, X_n$ to be less than $x, n-j$ of them to be greater than x, and 1 of them to equal x.
- Now, the probability density that any given set of $j-1$ of the X_i's are less than x, another given set of $n-j$ are all greater than x, and the remaining value is equal to x equals.

$$[F(x)]^{j-1}[1-F(x)^{n-j}]f(x)$$

Therefore,

$$\binom{n}{j-1, n-j, 1} = \frac{n!}{(n-j)!\,(j-1)!}$$

Order Statistics

independent

$$F_n(y_n) = P\{Y_n \le y_n\} = P\{Y_1 \le y_n, Y_2 \le y_n, \dots, Y_n \le y_n\} = [F(y_n)]^n$$

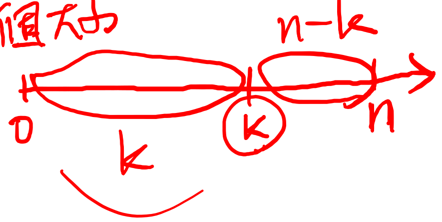
differentiation

$$f_n(y_n) = \frac{\partial F_n(y_n)}{\partial y} = n[F(y_n)]^n f(y_n)$$ ；令 $y = y_n$ 某一個常數 $\ge Y_1 \sim Y_n$

$$F_k(y) = P\{Y_k < y\} = \sum_{i=k}^{n} \binom{n}{i} (F(y))^i (1-F(y))^{n-i} = \sum_{i=k}^{n-1} \binom{n}{i} (F(y))^i (1-F(y))^{n-i} + (F(y))^n$$

$i=n$: $\binom{n}{n}(F(y))^n (1-F(y))^{n-n=0}$

第k個大的



Differentiation

$$f_k(y) = \frac{\partial F_k(y)}{\partial y} = \sum_{i=k}^{n-1} \binom{n}{i} i (F(y))^{i-1} f(y) (1-F(y))^{n-i} - \sum_{i=k}^{n-1} \binom{n}{i} (F(y))^i (n-i)(1-F(y))^{n-i-1} f(y) + n(F(y))^{n-1} f(y)$$

$$\Rightarrow \sum_{i=k}^{n-1} \frac{n!}{(i-1)!(n-i)!} (F(y))^{i-1} f(y)(1-F(y))^{n-i} - \sum_{i=k}^{n-1} \frac{n!}{i!(n-i-1)!} (F(y))^i (1-F(y))^{n-i-1} f(y) + n(F(y))^{n-1} f(y)$$

Order Statistics

$$\hookrightarrow \sum_{i=k}^{n-1} \frac{n!}{(i-1)!\,(n-i)!}(F(y))^{i-1}\cdot f(y)\,[1-F(y)]^{n-i} - \sum_{i=k}^{n-1}\frac{n!}{i!\,(n-i-1)!}(F(y))^{i}\,(1-F(y))^{n-i-1}\cdot f(y) + n(F(n))^{n-1} f(y)$$

if $i=k$: $$\frac{n!}{(k-1)!\,(n-k)!}(F(y))^{k-1}\cdot f(y)\,[1-F(y)]^{n-k} - \frac{n!}{k!\,(n-k-1)!}(F(y))^{k}\,(1-F(y))^{n-k-1}\cdot f(y) + n(F(n))^{n-1} f(y)$$

$i=k+1$: $$\frac{n!}{k!\,(n-k-1)!}(F(y))^{k}\cdot f(y)\,[1-F(y)]^{n-k-1} - \frac{n!}{(k+1)!\,(n-k-2)!}(F(y))^{k+1}\,(1-F(y))^{n-k-2}\cdot f(y) + n(F(n))^{n-1} f(y)$$

⋮

$i=n-1$: $$\frac{n!}{(n-2)!\,(n-n+1)!}(F(y))^{n-2}\cdot f(y)\,[1-F(y)]^{n-n+1} - \frac{n!}{(n-1)!\,(n-(n-1)-1)!}(F(y))^{n-1}\,(1-F(y))^{n-(n-1)-1}\cdot f(y)$$

$$\frac{n!}{(k-1)!\,(n-k)!}(F(y))^{k-1}\cdot f(y)[1-F(y)]^{n-k} - n\cdot(F(y))^{n-1}\cdot f(y) + n(F(n))^{n-1}\cdot f(y)$$

$$\rightarrow \frac{n!}{(k-1)!\,(n-k)!}(F(y))^{k-1}\cdot f(y)\,[1-F(y)]^{n-k}$$

Order Statistics

- Different partitions of the n random variables $X_1, \ldots X_n$ into the preceding three groups, it follows that the density function of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(n-j)!\,(j-1)!}[F(x)]^{j-1}[1 - F(x)^{n-j}]f(x)$$

Order Statistics

- **範例十二**

如果今天有一個樣本含有$2n+1$個random variables ($2n+1$ independent and identically distributed random variables)，則第$n+1$小的稱為sample median。假設今天樣本大小為3，且為uniform distribution over $(0,1)$，試問: sample median落在$\frac{1}{4}$與$\frac{3}{4}$之間的機率為何?

Order Statistics

Solution:

根據pp.51所導出來的公式得知$X_{(2)}$

$$f_{X_{(2)}}(x)=\frac{3!}{1!\,1!}x(1-x), 0<x<1$$

$Hence,$

$$P\left\{\frac{1}{4}<X_{(2)}<\frac{3}{4}\right\}=6\int_{1/4}^{3/4}x(1-x)\,dx=6\left\{\frac{x^2}{2}-\frac{x^3}{3}\right\}\Bigg|_{x=1/4}^{x=3/4}=\frac{11}{16}$$

Order Statistics

$The\ c.d.f.\ of\ X_{(j)}\ can\ be\ found\ by\ integrating\ pp.51\ formula$

$$F_{X_{(j)}}(y) = \frac{n!}{(n-j)!\,(j-1)!}\int_{-\infty}^{y}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x)dx$$

$$F_{X_{(j)}}(y) = P\{X_{(j)} \le y\} = P\{j\ or\ more\ of\ the\ X_i\ 's\ are \le y\}$$

$$= \sum_{k=j}^{n}\binom{n}{k}[F(y)]^k[1-F(y)]^{n-k}$$

$We\ take\ F\ to\ be\ the\ uniform\ (0,1)\ distribution\ [that\ is, f(x) = 1, 0 < x < 1],$
$then\ we\ obtain\ the\ interesting\ analytical\ identity$

$$\sum_{k=j}^{n}\binom{n}{k}y^k(1-y)^{n-k} = \frac{n!}{(n-j)!\,(j-1)!}\int_{0}^{y}x^{j-1}(1-x)^{n-j}dx\ , where\ 0 \le y \le 1$$

Order Statistics

Referring to *pp.51*

$$f_{X_{(j)}}(x) = \frac{n!}{(n-j)!\,(j-1)!}[F(x)]^{j-1}[1-F(x)^{n-j}]f(x)$$

We can obtain…

$$f_{X_{(i)},X_{(j)}}\left(x_i, x_j\right)$$
$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}[F(x_i)]^{i-1} \times \left[F\left(x_j\right) - F(x_i)\right]^{j-i-1}\left[1-F\left(x_j\right)\right]^{n-j} f(x_i) f\left(x_j\right)$$
$$for\ all\ x_i < x_j$$

Joint Probability Distribution of Functions of Random Variables

- Let X_1 and X_2 be jointly continuous random variables with joint probability density function $f_{X1,X2}$. It is sometimes necessary to obtain the joint distribution of the random variables Y_1 and Y_2, which arise as functions of X_1 and X_2. Specifically, suppose that $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ for some functions g_1 and g_2.

Joint Probability Distribution of Functions of Random Variables

- Assume that the functions g_1 and g_2 satisfy the following conditions:
  1. The equations $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ can be uniquely solved for x_1 and x_2 in terms of y_1 and y_2, with solutions given by, say, $x_1 = h_1(y_1, y_2)$, $x_2 = h_2(y_1, y_2)$.
  2. The functions g_1 and g_2 have continuous partial derivatives at all points (x_1, x_2) and are such that the 2×2 determinant.

Joint Probability Distribution of Functions of Random Variables

$$J(x_1,x_2)=\begin{vmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}\end{vmatrix}\equiv\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2}-\frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}\neq 0, at\ all\ point\ (x_1,x_2)$$

- Under these two conditions, it can be shown that the random variables Y_1 and Y_2 are jointly continuous with joint density function given by $f_{Y_1Y_2}(y_1,y_2)=f_{X_1,X_2}(x_1,x_2)|J(x_1,x_2)|^{-1}, where\ x_1=h_1(y_1,y_2), x_2=h_2(y_1,y_2)$
- A proof of above equation would proceed along the following lines

$$P\{Y_1\le y_1,Y_2\le y_2\}=\iint_{\substack{(x_1,x_2):\\ g_1(x_1,x_2)\le y_1\\ g_2(x_1,x_2)\le y_2}} f_{X_1,X_2}(x_1,x_2)dx_1dx_2$$

Joint Probability Distribution of Functions of Random Variables

- 範例十三

Let X_1 and X_2 be joint continuous random variables with probability density function f_{X_1,X_2}. Let $Y_1 = X_1 + X_2,\ \ Y_2 = X_1 - X_2$. Find the joint density function of Y_1 and Y_2 in terms of f_{X_1,X_2}.

Solution:

$$Let\ g_1(x_1, x_2) = x_1 + x_2\ and\ g_2(x_1, x_2) = x_1 - x_2.$$

$$J(x_1, x_2) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Joint Probability Distribution of Functions of Random Variables

Also, since the equations $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ have $x_1 = (y_1 + y_2)/2, x_2 = (y_1 - y_2)/2$ as their solution, it follows from

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2)|J(x_1, x_2)|^{-1}, where J(x_1, x_2) = -2$$

Then

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2} f_{X_1, X_2}\left(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}\right)$$

For instance, if X_1 and X_2 are independent uniform (0,1) random variables, then

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{1}{2} & 0 \le y_1 + y_2 \le 2, 0 \le y_1 - y_2 \le 2 \\ 0 & otherwise \end{cases}$$

Joint Probability Distribution of Functions of Random Variables

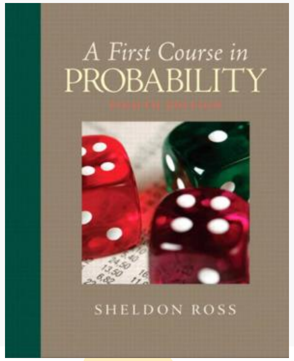
Or if X_1 and X_2 are independent **exponential random variables** with respective parameters λ_1 and λ_2, then

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} \frac{\lambda_1\lambda_2}{2}\exp\{-\lambda_1\left(\frac{y_1+y_2}{2}\right) - \lambda_2\left(\frac{y_1-y_2}{2}\right)\} & y_1+y_2 \geq 0, y_1-y_2 \geq 0 \\ 0 & otherwise \end{cases}$$

Finally, if X_1 and X_2 are independent **standard normal random variables**, then

$$f_{Y_1Y_2}(y_1, y_2) = \frac{1}{4\pi}e^{-\left[\frac{(y_1+y_2)^2}{8}+\frac{(y_1-y_2)^2}{8}\right]} = \frac{1}{4\pi}e^{-\frac{(y_1^2+y_2^2)}{4}} = \frac{1}{\sqrt{4\pi}}e^{-\frac{y_1^2}{4}}\frac{1}{\sqrt{4\pi}}e^{-\frac{y_2^2}{4}}$$

[#12] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

6.8. The joint probability density function of X and Y is given by

$$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y,\ 0 < y < \infty$$

(a) Find c.
(b) Find the marginal densities of X and Y.
(c) Find $E[X]$.

6.9. The joint probability density function of X and Y is given by

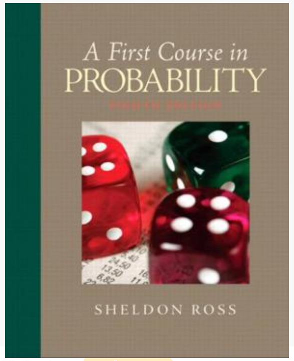
$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$

(a) Verify that this is indeed a joint density function.
(b) Compute the density function of X.
(c) Find $P\{X > Y\}$.
(d) Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.
(e) Find $E[X]$.
(f) Find $E[Y]$.

6.12. The number of people that enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#12] Assignment



6.40. The joint probability mass function of X and Y is given by

$$p(1,1) = \frac{1}{8} \qquad p(1,2) = \frac{1}{4}$$
$$p(2,1) = \frac{1}{8} \qquad p(2,2) = \frac{1}{2}$$

(a) Compute the conditional mass function of X given $Y = i, i = 1, 2$.

(b) Are X and Y independent?

(c) Compute $P\{XY \leq 3\}, P\{X + Y > 2\}, P\{X/Y > 1\}$.

6.41. The joint density function of X and Y is given by

$$f(x, y) = xe^{-x(y+1)} \quad x > 0, \ y > 0$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.

(b) Find the density function of $Z = XY$.

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))